

Advances in Intelligent Systems and Computing 648

Patricia Melin

Oscar Castillo

Janusz Kacprzyk

Marek Reformat

William Melek *Editors*

# Fuzzy Logic in Intelligent System Design

Theory and Applications

 Springer

# **Advances in Intelligent Systems and Computing**

Volume 648

## **Series editor**

Janusz Kacprzyk, Polish Academy of Sciences, Warsaw, Poland  
e-mail: [kacprzyk@ibspan.waw.pl](mailto:kacprzyk@ibspan.waw.pl)

### *About this Series*

The series “Advances in Intelligent Systems and Computing” contains publications on theory, applications, and design methods of Intelligent Systems and Intelligent Computing. Virtually all disciplines such as engineering, natural sciences, computer and information science, ICT, economics, business, e-commerce, environment, healthcare, life science are covered. The list of topics spans all the areas of modern intelligent systems and computing.

The publications within “Advances in Intelligent Systems and Computing” are primarily textbooks and proceedings of important conferences, symposia and congresses. They cover significant recent developments in the field, both of a foundational and applicable character. An important characteristic feature of the series is the short publication time and world-wide distribution. This permits a rapid and broad dissemination of research results.

### *Advisory Board*

#### Chairman

Nikhil R. Pal, Indian Statistical Institute, Kolkata, India  
e-mail: [nikhil@isical.ac.in](mailto:nikhil@isical.ac.in)

#### Members

Rafael Bello Perez, Universidad Central “Marta Abreu” de Las Villas, Santa Clara, Cuba  
e-mail: [rbellop@uclv.edu.cu](mailto:rbellop@uclv.edu.cu)

Emilio S. Corchado, University of Salamanca, Salamanca, Spain  
e-mail: [escorchado@usal.es](mailto:escorchado@usal.es)

Hani Hagra, University of Essex, Colchester, UK  
e-mail: [hani@essex.ac.uk](mailto:hani@essex.ac.uk)

László T. Kóczy, Széchenyi István University, Győr, Hungary  
e-mail: [koczy@sze.hu](mailto:koczy@sze.hu)

Vladik Kreinovich, University of Texas at El Paso, El Paso, USA  
e-mail: [vladik@utep.edu](mailto:vladik@utep.edu)

Chin-Teng Lin, National Chiao Tung University, Hsinchu, Taiwan  
e-mail: [ctlin@mail.nctu.edu.tw](mailto:ctlin@mail.nctu.edu.tw)

Jie Lu, University of Technology, Sydney, Australia  
e-mail: [Jie.Lu@uts.edu.au](mailto:Jie.Lu@uts.edu.au)

Patricia Melin, Tijuana Institute of Technology, Tijuana, Mexico  
e-mail: [epmelin@hafsamx.org](mailto:epmelin@hafsamx.org)

Nadia Nedjah, State University of Rio de Janeiro, Rio de Janeiro, Brazil  
e-mail: [nadia@eng.uerj.br](mailto:nadia@eng.uerj.br)

Ngoc Thanh Nguyen, Wrocław University of Technology, Wrocław, Poland  
e-mail: [Ngoc-Thanh.Nguyen@pwr.edu.pl](mailto:Ngoc-Thanh.Nguyen@pwr.edu.pl)

Jun Wang, The Chinese University of Hong Kong, Shatin, Hong Kong  
e-mail: [jwang@mae.cuhk.edu.hk](mailto:jwang@mae.cuhk.edu.hk)

More information about this series at <http://www.springer.com/series/11156>

Patricia Melin · Oscar Castillo  
Janusz Kacprzyk · Marek Reformat  
William Melek  
Editors

# Fuzzy Logic in Intelligent System Design

Theory and Applications

 Springer

*Editors*

Patricia Melin  
Division of Graduate Studies and Research  
Tijuana Institute of Technology  
Tijuana, Baja California  
Mexico

Marek Reformat  
Department of Electrical and Computer  
Engineering  
University of Alberta  
Edmonton, AB  
Canada

Oscar Castillo  
Division of Graduate Studies and Research  
Tijuana Institute of Technology  
Tijuana, Baja California  
Mexico

William Melek  
Laboratory of Computational Intelligence  
and Automation  
University of Waterloo  
Waterloo, ON  
Canada

Janusz Kacprzyk  
Systems Research Institute  
Polish Academy of Sciences  
Warsaw  
Poland

ISSN 2194-5357                      ISSN 2194-5365 (electronic)  
Advances in Intelligent Systems and Computing  
ISBN 978-3-319-67136-9              ISBN 978-3-319-67137-6 (eBook)  
DOI 10.1007/978-3-319-67137-6

Library of Congress Control Number: 2017952851

© Springer International Publishing AG 2018

This work is subject to copyright. All rights are reserved by the Publisher, whether the whole or part of the material is concerned, specifically the rights of translation, reprinting, reuse of illustrations, recitation, broadcasting, reproduction on microfilms or in any other physical way, and transmission or information storage and retrieval, electronic adaptation, computer software, or by similar or dissimilar methodology now known or hereafter developed.

The use of general descriptive names, registered names, trademarks, service marks, etc. in this publication does not imply, even in the absence of a specific statement, that such names are exempt from the relevant protective laws and regulations and therefore free for general use.

The publisher, the authors and the editors are safe to assume that the advice and information in this book are believed to be true and accurate at the date of publication. Neither the publisher nor the authors or the editors give a warranty, express or implied, with respect to the material contained herein or for any errors or omissions that may have been made. The publisher remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

Printed on acid-free paper

This Springer imprint is published by Springer Nature  
The registered company is Springer International Publishing AG  
The registered company address is: Gewerbestrasse 11, 6330 Cham, Switzerland

# Preface

We describe in this book recent advances on the use of fuzzy logic in design of hybrid intelligent systems based on nature-inspired optimization and their application in areas such as intelligent control and robotics, pattern recognition, medical diagnosis, time series prediction, and optimization of complex problems. The book is organized into nine main parts, which contain a group of papers around a similar subject. The first part consists of papers with the main theme of theoretical aspects of fuzzy logic, which basically consists of papers that propose new concepts and algorithms based on type-1 fuzzy systems. The second part contains papers with the main theme of type-2 fuzzy logic, which are basically papers dealing with new concepts and algorithms for type-2 fuzzy systems. The second part also contains papers describing applications of type-2 fuzzy systems in diverse areas, such as time series prediction and pattern recognition. The third part contains papers that present enhancements to meta-heuristics based on fuzzy logic techniques describing new nature-inspired optimization algorithms that use fuzzy dynamic adaptation of parameters. The fourth part presents emergent intelligent models, which range from quantum algorithms to cellular automata. The fifth part contains papers describing applications of fuzzy logic in diverse areas of medicine, such as diagnosis of hypertension and heart diseases. The sixth part contains papers describing new computational intelligence algorithms and their applications in different areas of intelligent control. The seventh part contains papers that present the use of fuzzy logic in different mathematic models. The eighth part deals with a diverse range of applications of fuzzy logic, ranging from environmental to autonomous navigation. The ninth part deals with theoretical concepts of fuzzy models.

In the first part of theoretical aspects of type-1 fuzzy logic, there are four papers that describe different contributions that propose new models, concepts, and algorithms centered on type-1 fuzzy systems. The aim of using fuzzy logic is to provide uncertainty management in modeling complex problems.

In the second part of type-2 fuzzy logic theory and applications, there are four papers that describe different contributions that propose new models, concepts, and algorithms centered on type-2 fuzzy systems. There are also papers that describe different contributions on the application of these kinds of type-2 fuzzy systems to

solve complex real-world problems, such as time series prediction, medical diagnosis, and pattern recognition.

In the third part of fuzzy logic for the augmentation of nature-inspired optimization meta-heuristics, there are six papers that describe different contributions that propose new models and concepts, which can be considered as the basis for enhancing nature-inspired algorithms with fuzzy logic. The aim of using fuzzy logic is to provide dynamic adaptation capabilities to the optimization algorithms, and this is illustrated with the cases of the bat algorithm, harmony search, and other methods. The nature-inspired methods include variations of ant colony optimization, particle swarm optimization, the bat algorithm, as well as new nature-inspired paradigms.

In the fourth part of emergent intelligent models, there are six papers that describe different contributions on the application of these kinds of models to solve complex real-world optimization problems, such as time series prediction, robotics, and pattern recognition.

In the fifth part of fuzzy logic applications in medicine, there are three papers that describe different contributions on the application of these kinds of fuzzy logic models to solve complex real-world problems, such as medical diagnosis.

In the sixth part of intelligent control, there are six papers that describe different contributions that propose new models, concepts, and algorithms for designing intelligent controllers for different plants. The aim of using these algorithms is to provide methods and solution to some real-world problem control areas, such as scheduling, planning, and robotics.

In the seventh part, there are five papers that are presenting the application of fuzzy logic in different mathematical models. There are also papers that describe different contributions on the application of these kinds of fuzzy models to solve complex real-world problems, such as in intelligent control.

In the eighth part, there are four papers dealing with applications of fuzzy logic, like in diagnosing air quality or vehicle navigation. In addition, theoretical contributions are presented in regard to how we can apply fuzzy logic.

Finally, in the ninth part, there are six papers presenting theoretical concepts of fuzzy models. The concepts range from fuzzy linear programming to fuzzy restricted Boltzmann machines.

In conclusion, the edited book comprises papers on diverse aspects of fuzzy logic, neural networks, and nature-inspired optimization meta-heuristics and their application in areas such as intelligent control and robotics, pattern recognition, time series prediction, and optimization of complex problems. There are theoretical aspects as well as application papers.

June 2017

Patricia Melin  
Oscar Castillo  
Janusz Kacprzyk  
Marek Reformat  
William Melek

# Contents

## Theoretical Aspects of Fuzzy Logic

<b>Can Multi-constraint Fuzzy Optimization <i>Bring Complex Problems in Selecting Optimal Solar Power Generating System into Focus?</i> . . . . .</b>	<b>3</b>
Akash Dand, Chetankumar Patil, and Ashok Deshpande	

<b>Relating Fuzzy Set Similarity Measures . . . . .</b>	<b>9</b>
Valerie Cross	

<b>Correlation Measures for Bipolar Rating Profiles . . . . .</b>	<b>22</b>
Fernando Monroy-Tenorio, Ildar Batyrshin, Alexander Gelbukh, Valerie Solovyev, Nailya Kubysheva, and Imre Rudas	

<b>Solving Real-World Fuzzy Quadratic Programming Problems by Dual Parametric Approach . . . . .</b>	<b>33</b>
Ricardo Coelho	

## Type-2 Fuzzy Logic

<b>A Type-2 Fuzzy Hybrid Expert System for Commercial Burglary . . . . .</b>	<b>41</b>
M.H. Fazel Zarandi, A. Seifi, H. Esmaeeli, and Sh. Sotudian	

<b>A Type-2 Fuzzy Expert System for Diagnosis of Leukemia . . . . .</b>	<b>52</b>
Ali Akbar Sadat Asl and Mohammad Hossein Fazel Zarandi	

<b>Comparative Study of Metrics That Affect in the Performance of the Bee Colony Optimization Algorithm Through Interval Type-2 Fuzzy Logic Systems . . . . .</b>	<b>61</b>
Leticia Amador-Angulo and Oscar Castillo	

<b>Type-2 Fuzzy Approach in Multi Attribute Group Decision Making Problem . . . . .</b>	<b>73</b>
Zohre Moattar Husseini and Mohammad Hossein Fazel Zarandi	



## **Fuzzy Logic in Metaheuristics**

<b>A New Approach for Dynamic Mutation Parameter in the Differential Evolution Algorithm Using Fuzzy Logic . . . . .</b>	<b>85</b>
--	-----------

Patricia Ochoa, Oscar Castillo, and José Soria

<b>Study on the Use of Type-1 and Interval Type-2 Fuzzy Systems Applied to Benchmark Functions Using the Fuzzy Harmony Search Algorithm . . . . .</b>	<b>94</b>
---	-----------

Cinthia Peraza, Fevrier Valdez, and Oscar Castillo

<b>Fuzzy Adaptation for Particle Swarm Optimization for Modular Neural Networks Applied to Iris Recognition . . . . .</b>	<b>104</b>
---	------------

Daniela Sánchez, Patricia Melin, and Oscar Castillo

<b>A New Metaheuristic Based on the Self-defense Mechanisms of the Plants with a Fuzzy Approach Applied to the CEC2015 Functions . . . . .</b>	<b>115</b>
--	------------

Camilo Caraveo, Fevrier Valdez, and Oscar Castillo

<b>Fuzzy Chemical Reaction Algorithm with Dynamic Adaptation of Parameters . . . . .</b>	<b>122</b>
--	------------

David de la O, Oscar Castillo, Leslie Astudillo, and Jose Soria

<b>Methodology for the Optimization of a Fuzzy Controller Using a Bio-inspired Algorithm . . . . .</b>	<b>131</b>
--	------------

Marylu L. Lagunes, Oscar Castillo, and Jose Soria

## **Emergent Intelligent Models**

<b>Cellular Automata Enhanced Quantum Inspired Edge Detection . . . . .</b>	<b>141</b>
---	------------

Yoshio Rubio, Oscar Montiel, and Roberto Sepúlveda

<b>Competitive Hybrid Ensemble Using Neural Network and Decision Tree . . . . .</b>	<b>147</b>
---	------------

Davin Kaing and Larry Medsker

<b>Speeding Up Quantum Genetic Algorithms in Matlab Through the Quack_GPU V1 . . . . .</b>	<b>156</b>
--	------------

Oscar Montiel, Roberto Sepúlveda, and Yoshio Rubio

<b>Evolving Granular Fuzzy Min-Max Regression . . . . .</b>	<b>162</b>
---	------------

Alisson Porto and Fernando Gomide

<b>Optimization of Deep Neural Network for Recognition with Human Iris Biometric Measure . . . . .</b>	<b>172</b>
--	------------

Fernando Gaxiola, Patricia Melin, Fevrier Valdez, and Juan R. Castro

<b>Dynamic Local Trend Associations in Analysis of Comovements of Financial Time Series . . . . .</b>	181
Francisco Javier García-López, Ildar Batyrshin, and Alexander Gelbukh	
<b>Fuzzy Logic in Medicine</b>	
<b>An Expert System Based on Fuzzy Bayesian Network for Heart Disease Diagnosis . . . . .</b>	191
M.H. Fazel Zarandi, A. Seifi, M.M. Ershadi, and H. Esmaeeli	
<b>A Hybrid Intelligent System Model for Hypertension Risk Diagnosis . . . . .</b>	202
Ivette Miramontes, Gabriela Martínez, Patricia Melin, and German Prado-Arechiga	
<b>Estimation of Population Pharmacokinetic Model Parameters Using a Genetic Algorithm . . . . .</b>	214
Carlos Sepúlveda, Oscar Montiel, José M. Cornejo, and Roberto Sepúlveda	
<b>Intelligent Control</b>	
<b>Outdoor Robot Navigation Based on Particle Swarm Optimization . . . . .</b>	225
Erasmus Gabriel Martínez Soltero, Carlos López-Franco, Alma Y. Alanis, and Nancy Arana-Daniel	
<b>Trajectory Optimization for an Autonomous Mobile Robot Using the Bat Algorithm . . . . .</b>	232
Jonathan Perez, Patricia Melin, Oscar Castillo, Fevrier Valdez, Claudia Gonzalez, and Gabriela Martinez	
<b>Neural Identifier-Control Scheme for Nonlinear Discrete Systems with Input Delay . . . . .</b>	242
Jorge D. Rios, Alma Y. Alanís, Nancy Arana-Daniel, and Carlos López-Franco	
<b>An Application of Neural Network to Heavy Oil Distillation with Recognitions with Intuitionistic Fuzzy Estimation . . . . .</b>	248
Sotir Sotirov, Evdokia Sotirova, Dicho Stratiev, Danail Stratiev, and Nikolay Sotirov	
<b>PID Implemented by a Type-1 Fuzzy Logic System with Back-Propagation Algorithm for Online Tuning of Its Gains . . . . .</b>	256
Alberto Álvarez, David Reyes, Ernesto J. Rincón, José Valderrama, Pascual Noradino, and Gerardo M. Méndez	

<b>A PID Using a Non-singleton Fuzzy Logic System Type 1 to Control a Second-Order System</b> . . . . .	264
David Reyes, Alberto Álvarez, Ernesto J. Rincón, José Valderrama, Pascual Noradino, and Gerardo M. Méndez	
<b>Fuzzy Multi-Criteria Decision Making and Fuzzy Information Gain Based Automotive Recommender System</b> . . . . .	270
Charu Gupta and Amita Jain	
<b>Fuzzy Logic in Mathematics</b>	
<b>The Shape of the Optimal Value of a Fuzzy Linear Programming Problem</b> . . . . .	281
Milan Hladík and Michal Černý	
<b>How to Gauge the Accuracy of Fuzzy Control Recommendations: A Simple Idea</b> . . . . .	287
Patricia Melin, Oscar Castillo, Andrzej Pownuk, Olga Kosheleva, and Vladik Kreinovich	
<b>“On-the-fly” Parameter Identification for Dynamic Systems Control, Using Interval Computations and Reduced-Order Modeling</b> . . . . .	293
Leobardo Valera, Angel Garcia Contreras, and Martine Ceberio	
<b>Normalization-Invariant Fuzzy Logic Operations Explain Empirical Success of Student Distributions in Describing Measurement Uncertainty</b> . . . . .	300
Hamza Alkhatib, Boris Kargoll, Ingo Neumann, and Vladik Kreinovich	
<b>Can We Detect Crisp Sets Based Only on the Subsethood Ordering of Fuzzy Sets? Fuzzy Sets and/or Crisp Sets Based on Subsethood of Interval-Valued Fuzzy Sets?</b> . . . . .	307
Christian Servin, Gerardo Muela, and Vladik Kreinovich	
<b>Applications of Fuzzy Logic</b>	
<b>Two Hybrid Expert System for Diagnosis Air Quality Index (AQI)</b> . . . .	315
Leila Abdolkarimzadeh, Milad Azadpour, and M.H. Fazel Zarandi	
<b>Fuzzy Rule Based Expert System to Diagnose Chronic Kidney Disease</b> . . . . .	323
M.H. Fazel Zarandi and Mona Abdolkarimzadeh	
<b>A Theory of Event Possibility with Application to Vehicle Waypoint Navigation</b> . . . . .	329
Daniel G. Schwartz	
<b>Intuitionistic Fuzzy Functional Differential Equations</b> . . . . .	335
Bouchra Ben Amma, Said Melliani, and L.S. Chadli	

**Theoretical Concepts of Fuzzy Models**

<b>Defects in the Defuzzification of Periodic Membership Functions on Orthogonal Coordinates and a Solution</b> . . . . .	361
Takashi Mitsuishi	
<b>Taking into Account Interval (and Fuzzy) Uncertainty Can Lead to More Adequate Statistical Estimates</b> . . . . .	371
Ligang Sun, Hani Dbouk, Ingo Neumann, Steffen Schön, and Vladik Kreinovich	
<b>Weak and Strong Solutions for Fuzzy Linear Programming Problems</b> . . . . .	382
Juan Carlos Figueroa-García and Germán Hernández-Peréz	
<b>Fuzzy Restricted Boltzmann Machines</b> . . . . .	392
Robert W. Harrison and Christopher Freas	
<b>Exotic Semirings and Uncertainty</b> . . . . .	399
Mark J. Wierman	
<b>Restricted Equivalence Function on <math>L([0, 1])</math></b> . . . . .	410
Eduardo S. Palmeira and Benjamín Bedregal	
<b>Author Index</b> . . . . .	421

# Can We Detect Crisp Sets Based Only on the Subsethood Ordering of Fuzzy Sets? Fuzzy Sets and/or Crisp Sets Based on Subsethood of Interval-Valued Fuzzy Sets?

Christian Servin<sup>1(✉)</sup>, Gerardo Muela<sup>2</sup>, and Vladik Kreinovich<sup>2</sup>

<sup>1</sup> Computer Science and Information Technology Systems Department,  
El Paso Community College, 919 Hunter, El Paso, TX 79915, USA  
cservin@gmail.com

<sup>2</sup> Department of Computer Science, University of Texas at El Paso,  
500 W. University, El Paso, TX 79968, USA  
gdmuela@miners.utep.edu, vladik@utep.edu

**Abstract.** Fuzzy sets are naturally ordered by the subsethood relation  $A \subseteq B$ . If we only know which set which fuzzy set is a subset of which – and have no access to the actual values of the corresponding membership functions – can we detect which fuzzy sets are crisp? In this paper, we show that this is indeed possible. We also show that if we start with interval-valued fuzzy sets, then we can similarly detect type-1 fuzzy sets and crisp sets.

## 1 Formulation of the Problem

**Fuzzy Sets: A Brief Reminder.** A *fuzzy set* is usually defined as a function  $\mu : U \rightarrow [0, 1]$  from some set  $U$  (called *Universe of discourse*) to the interval  $[0, 1]$ ; see, e.g., [1–3]. This function is also known as a *membership function*.

A fuzzy set  $A$  with a membership function  $\mu_A(x)$  is called a *subset* of a fuzzy set  $B$  with a membership function  $\mu_B(x)$  if  $\mu_A(x) \leq \mu_B(x)$  for all  $x$ . The subsethood relation is an *order* in the sense that it is reflexive ( $A \subseteq A$ ), asymmetric ( $A \subseteq B$  and  $B \subseteq A$  imply  $A = B$ ), and transitive ( $A \subseteq B$  and  $B \subseteq C$  imply  $A \subseteq C$ ).

Traditional (*crisp*) sets  $S$  can be viewed as particular cases of fuzzy sets, with their characteristic functions playing the role of membership functions:  $\mu_S(x) = 1$  if  $x \in S$  and  $\mu_S(x) = 0$  if  $x \notin S$ .

**A Natural Question: Can We Detect Crisp Sets Based Only on the Subsethood Ordering of Fuzzy Sets?** If we have a class  $F$  of all fuzzy sets, and for each fuzzy set  $A$  and for each element  $x \in U$ , we know the value  $\mu_A(x)$  of the corresponding membership function, then we can easily detect which of the fuzzy sets are crisp: a fuzzy set is crisp if for every  $x \in U$ , we have either  $\mu_A(x) = 0$  or  $\mu_A(x) = 1$ .

Suppose now that we have a class  $F$  of all fuzzy sets with the subethood ordering  $A \subseteq B$  – but we have no access to the actual values of the corresponding membership functions. Based only on this ordering relation  $A \subseteq B$ , can we then detect crisp sets?

**What If We Only Consider Interval-Valued Fuzzy Sets.** A similar question can be asked if we consider interval-valued fuzzy sets, for which the value of the membership function is a subinterval of the interval  $[0, 1]$ :  $\mu(x) = [\underline{\mu}(x), \overline{\mu}(x)] \subseteq [0, 1]$ , and  $A \subseteq B$  means that  $\underline{\mu}_A(x) \leq \underline{\mu}_B(x)$  and  $\overline{\mu}_A(x) \leq \overline{\mu}_B(x)$  for all  $x$ .

**What We Do in This Paper.** In this paper, we prove that in both cases – when we consider fuzzy sets and when we consider interval-valued fuzzy sets – we can indeed detect crisp sets and type-1 fuzzy sets based only on the subethood relation  $A \subseteq B$ .

## 2 What If We Consider $[0, 1]$ -Based Fuzzy Sets

**Our Plan.** To describe crisp sets in terms of the subethood relation  $A \subseteq B$ , we will follow the following four steps:

- first, we will prove that the empty set  $\emptyset$  can be uniquely determined based on the subethood relation;
- second, we will show that 1-element crisp sets, i.e., sets of the type  $\{x_0\}$ , can be thus determined,
- third, we will prove that 1-element fuzzy sets, i.e., fuzzy sets  $A$  for which for some  $x_0 \in U$ , we have  $\mu_A(x_0) > 0$  and  $\mu_A(x) = 0$  for all  $x \neq x_0$ , can be determined based on the subethood relation, and
- finally, we prove that crisp sets can be uniquely determined based on the subethood relation.

**First Step: How to Detect an Empty Set?** An empty set  $\emptyset$  is a fuzzy set for which  $\mu_\emptyset(x) = 0$  for all  $x \in U$ . The detection of an empty set can be made based on the following simple result:

**Proposition 1.** *A fuzzy set  $A$  is an empty set if and only if  $A \subseteq B$  for all fuzzy sets  $B$ .*

**Proof.**

1°. Let us first prove that when  $A = \emptyset$ , then  $A \subseteq B$  for all fuzzy sets  $B$ .

Indeed, for every fuzzy set  $B$ , we have  $0 \leq \mu_B(x)$  for all  $x$  and thus,  $\mu_\emptyset(x) = 0 \leq \mu_B(x)$  for all  $x$ , i.e., we indeed have  $\emptyset \subseteq B$ .

2°. Let us now prove that, vice versa, if for some fuzzy set  $A$ , we have  $A \subseteq B$  for every possible fuzzy set  $B$ , then  $A = \emptyset$ .

Indeed, in particular, the property  $A \subseteq B$  is true for the case when  $B$  is the empty set. In this case, from the fact that  $\mu_A(x) \leq \mu_B(x) = \mu_\emptyset(x) = 0$ , we conclude that  $\mu_A(x) = 0$  for all  $x$ , i.e., that  $A$  is indeed the empty set.

The proposition is proven.

**Second Step: How to Detect 1-element Crisp Sets Based on the Subsethood Relation.** Let us prove the following auxiliary result.

**Proposition 2.** *A non-empty fuzzy set  $A$  is a one-element crisp set if and only if the following two conditions are satisfied:*

- *the class  $\{B : B \subseteq A\}$  is linearly ordered and*
- *for no proper superset  $A'$  of  $A$ , the class  $\{B : B \subseteq A'\}$  is linearly ordered.*

**Proof.**

1°. Let us first prove that every 1-element crisp set, i.e., every set of the type  $A = \{x_0\}$ , satisfies the above two properties.

1.1°. Let us prove the first property: that the class  $\{B : B \subseteq A\}$  is linearly ordered.

Indeed, for the given set  $A$ , we have  $\mu_A(x_0) = 1$  and  $\mu_A(x) = 0$  for all  $x \neq x_0$ . So, if  $B \subseteq A$ , i.e., if  $\mu_B(x) \leq \mu_A(x)$  for all  $x$ , this means that  $\mu_B(x) = 0$  for all  $x \neq x_0$ . Thus, for such sets  $B$ , the only non-zero value of the membership function may be attained when  $x = x_0$ .

So, if we have two sets  $B \subseteq A$  and  $B' \subseteq A$ , then for these two sets,  $\mu_B(x) = \mu_{B'}(x) = 0$  for all  $x \neq x_0$ . Thus:

- if  $\mu_B(x_0) \leq \mu_{B'}(x_0)$ , then, as one can easily check, we have  $\mu_B(x) \leq \mu_{B'}(x)$  for all  $x$ , i.e. we have  $B \subseteq B'$ , and
- if  $\mu_{B'}(x_0) \leq \mu_B(x_0)$ , then, as one can easily check, we have  $\mu_{B'}(x) \leq \mu_B(x)$  for all  $x$ , we have  $B' \subseteq B$ .

Thus, for every two fuzzy sets  $B$  and  $B'$  from the class  $\{B : B \subseteq A\}$ , we have either  $B \subseteq B'$  or  $B' \subseteq B$ . So, this class is indeed linearly ordered.

1.2°. Let us now prove that no proper superset  $A'$  of the 1-element set  $A = \{x_0\}$  has the property that the class  $\{B : B \subseteq A'\}$  is linearly ordered.

For the set  $A = \{x_0\}$ , we have  $\mu_A(x_0) = 1$  and  $\mu_A(x) = 0$  for all other  $x$ . If  $A'$  is a superset of  $A$ , this means that  $\mu_{A'}(x) = 1$ . The fact that  $A'$  is a proper superset means that  $A' \neq A$ , thus we have  $\mu_{A'}(x') > 0$  for some  $x' \neq x_0$ . In this case, we can define the following fuzzy set  $B$ :  $\mu_B(x') = \mu_{A'}(x')$  and  $\mu_B(x) = 0$  for all  $x \neq x_0$ . Then, we have  $B \subseteq A'$ ,  $A \subseteq A'$ , but  $B \not\subseteq A$  (since  $\mu_B(x') > 0$  and thus,  $\mu_B(x') \not\leq \mu_A(x') = 0$ ) and  $A \not\subseteq B$  (since  $1 = \mu_A(x_0) \not\leq \mu_B(x_0) = 0$ ). Thus, the class  $\{B : B \subseteq A'\}$  is indeed not linearly ordered.

2°. Let us prove that, vice versa, if a fuzzy set  $A$  has the above two properties, then it is a one-element crisp set.

2.1°. Let us first prove, by contradiction, that we can only have one element  $x$  for which  $\mu_A(x) > 0$ . Indeed. if  $\mu_A(x_1) > 0$  and  $\mu_A(x_2) > 0$  for some  $x_1 \neq x_2$ , then we can take the following fuzzy sets  $B_1$  and  $B_2$ :

- $\mu_{B_1}(x_1) = \mu_A(x_1)$  and  $\mu_{B_1}(x) = 0$  for all other  $x$ , and
- $\mu_{B_2}(x_2) = \mu_A(x_2)$  and  $\mu_{B_2}(x) = 0$  for all other  $x$ .

Here,  $B_1 \subseteq A$  and  $B_2 \subseteq A$ , but  $B_2 \not\subseteq B_1$  and  $B_1 \not\subseteq B_2$  – which contradicts to our assumption that the class  $\{B : B \subseteq A\}$  is linearly ordered.

2.2°. Due to Part 2.1, we have  $\mu_A(x_0) > 0$  for at most one element  $x_0$ ; for all  $x \neq x_0$ , we have  $\mu_A(x) = 0$ . Let us prove, by contradiction, that  $\mu_A(x_0) = 1$ , i.e., that  $A$  is indeed a one-element crisp set.

Indeed, if  $\mu_A(x_0) < 1$ , then we can consider the following proper superset  $A' \supseteq A$ :  $\mu_{A'}(x_0) = (1 + \mu_A(x_0))/2 < 1$  and  $\mu_{A'}(x) = 0$  for all other  $x$ . Similarly to Part 1.1 of this proof, we can prove that for this superset  $A'$ , the class  $\{B : B \subseteq A'\}$  is linearly ordered – which contradicts to our assumption that such a proper superset does not exist.

The proposition is proven.

**Third Step: How to Detect 1-element Fuzzy Sets Based on the Subsethood Relation.** We say that a fuzzy set is a *1-element set* if for some  $x_0 \in X$ , we have  $\mu_A(x_0) > 0$  and  $\mu_A(x) = 0$  for all  $x \neq x_0$ . Let us prove the following auxiliary result.

**Proposition 3.** *A non-empty fuzzy set  $A$  is a one-element fuzzy set if and only if the class  $\{B : B \subseteq A\}$  is linearly ordered.*

**Proof.**

1°. Arguments similar to Part 1.1 of the proof of Proposition 2 show that if  $A$  is a one-element fuzzy set, then the class  $\{B : B \subseteq A\}$  is linearly ordered.

2°. Vice versa, if  $A$  is not an empty set and not a one-element fuzzy set, this means that there exist at least two values  $x_1 \neq x_2$  for which  $\mu_A(x_1) > 0$  and  $\mu_A(x_2) > 0$ . We can then take the following fuzzy sets  $B_1$  and  $B_2$ :

- $\mu_{B_1}(x_1) = \mu_A(x_1)$  and  $\mu_{B_1}(x) = 0$  for all  $x \neq x_1$ , and
- $\mu_{B_2}(x_2) = \mu_A(x_2)$  and  $\mu_{B_2}(x) = 0$  for all  $x \neq x_2$ .

Then  $B_1 \subseteq A$  and  $B_2 \subseteq A$ , but  $B_1 \not\subseteq B_2$  and  $B_2 \not\subseteq B_1$ . Thus, the class  $\{B : B \subseteq A\}$  is not linearly ordered.

The proposition is proven.

**Final Result: How to Detect Crisp Sets Based on the Subsethood Relation.** Let us prove the following auxiliary result.

**Theorem 1.** *A fuzzy set  $A$  is crisp if and only if every one-element fuzzy subset  $B \subseteq A$  can be embedded in a one-element crisp subset of  $A$ .*

*Comment.* In other words,

$$A \text{ is crisp} \Leftrightarrow \forall A (B \text{ is a one-element fuzzy subset of } A \Rightarrow \exists C ((B \subseteq C \subseteq A) \& (C \text{ is a 1-element crisp set}))).$$



**Proof.**

1°. Let  $A$  be a crisp set, and let  $B \subseteq A$  be a 1-element fuzzy set. By definition, this means that for some  $x_0$ , we have  $\mu_B(x_0) > 0$  and  $\mu_B(x) = 0$  for all other  $x$ .

Since the set  $A$  is crisp, the only possible values of  $\mu_A(x_0)$  are 0 and 1. From  $\mu_B(x_0) \leq \mu_A(x_0)$ , we conclude that  $\mu_A(x_0) > 0$  and thus, that  $\mu_A(x_0) = 1$ . So,  $x_0 \in A$  and hence  $B \subseteq \{x_0\} \subseteq A$ .

2°. Vice versa, if  $A$  is not a crisp set, this means that for some element  $x_0$ , we have  $0 < \mu_A(x_0) < 1$ . In this case, we can take the following 1-element fuzzy set  $B \subseteq A$ :  $\mu_B(x_0) = \mu_A(x_0)$  and  $\mu_B(x) = 0$  for all  $x \neq x_0$ . Here,  $B \subseteq A$ , but the only 1-element crisp set  $C$  containing  $B$  is the set  $C = \{x_0\}$ , and this 1-element crisp set is *not* a subset of the original set  $A$ :  $C \not\subseteq A$ .

The theorem is proven.

### 3 What If We Consider Interval-Valued Fuzzy Sets

**First Step: How to Detect an Empty Set.** An empty set  $\emptyset$  is an interval-valued fuzzy set for which  $\mu_\emptyset(x) = [0, 0]$  for all  $x \in U$ . The detection of an empty set can be made based on the following result:

**Proposition 4.** *An interval-valued fuzzy set  $A$  is an empty set if and only if  $A \subseteq B$  for all interval-valued fuzzy sets  $B$ .*

**Proof** is similar to proof of Proposition 1.

**Second Step: How to Detect Special 1-element Interval-Valued Fuzzy Sets Based on the Subsethood Relation.** Let's introduce an auxiliary notion. We say that an interval-valued fuzzy set  $A$  is *special* if for some element  $x_0$ , we have  $\mu_A(x_0) = [0, a]$  for some number  $a > 0$  and  $\mu_A(x) = 0$  for all  $x \neq x_0$ .

**Proposition 5.** *A non-empty interval-valued fuzzy set  $A$  is special if and only if the class  $\{B : B \subseteq A\}$  is linearly ordered.*

**Proof.**

1°. For special sets (in the sense of the above definition), the fact that the class  $\{B : B \subseteq A\}$  is linearly ordered can be proven similarly to Part 1.1 of the proof of Proposition 2.

2°. Let us now prove that, vice versa, if for some non-empty interval-valued fuzzy set  $A$ , the class  $\{B : B \subseteq A\}$  is linearly ordered, then the set  $A$  is special.

2.1°. Since  $A$  is non-empty, there exists an element  $x_0$  for which  $\mu_A(x_0) \neq [0, 0]$ . Let us prove, by contradiction, that for every other element  $x \neq x_0$ , we have  $\mu_A(x) = [0, 0]$ .

Indeed, if we had  $\mu_A(x_1) \neq [0, 0]$  for some  $x_1 \neq x_0$ , then we would be able to take the following two sets  $B_0$  and  $B_1$ :

- $\mu_{B_0}(x_0) = \mu_A(x_0)$  and  $\mu_{B_0}(x) = [0, 0]$  for all  $x \neq x_0$ , and
- $\mu_{B_1}(x_1) = \mu_A(x_1)$  and  $\mu_{B_1}(x) = [0, 0]$  for all  $x \neq x_1$ .

In this case,  $B_0 \subseteq A$  and  $B_1 \subseteq A$ , but  $B_0 \not\subseteq B_1$  and  $B_1 \not\subseteq B_0$ . This contradicts our assumption that the class  $\{B : B \subseteq A\}$  is linearly ordered.

2.2°. To complete the proof of the proposition, we need to prove that the value  $\mu_A(x_0) = [\underline{\mu}_A(x_0), \bar{\mu}_A(x_0)]$  has the form  $[0, a]$  for some  $a > 0$ , i.e., that  $\underline{\mu}_A(x_0) = 0$ .

We will prove it by contradiction. Suppose that, vice versa,  $\underline{\mu}_A(x_0) > 0$ . In this case, we can take the following sets  $B_1$  and  $B_2$ :

- $\mu_{B_1}(x_0) = [0.5 \cdot \underline{\mu}_A(x_0), 0.5 \cdot \underline{\mu}_A(x_0)]$  and  $\mu_{B_1}(x) = 0$  for all  $x \neq x_0$ , and
- $\mu_{B_2}(x_0) = [0, \underline{\mu}_A(x_0)]$  and  $\mu_{B_2}(x) = 0$  for all  $x \neq x_0$ .

Then,  $B_1 \subseteq A$  and  $B_2 \subseteq A$ , but  $B_1 \not\subseteq B_2$  and  $B_2 \not\subseteq B_1$ . This contradicts our assumption that the class  $\{B : B \subseteq A\}$  is linearly ordered.

The proposition is proven.

**Third Step: How to Detect 1-element type-1 Fuzzy Sets Based on the Subsethood Relation.** We say that an interval-valued fuzzy set is a *1-element type-1* fuzzy set if there exists an element  $x_0$  for which  $\mu_A(x_0) = [a, a]$  for some  $a > 0$  and  $\mu_A(x) = [0, 0]$  for all  $x \neq x_0$ .

**Proposition 6.** *A non-empty interval-valued fuzzy set  $A$  is a 1-element type-1 set if and only if it satisfies the following three properties:*

- *the set  $A$  is not special (in the sense of the above definition),*
- *there exists a special set  $B \subseteq A$  for which the class  $\{C : B \subseteq C \subseteq A\}$  is linearly ordered, and*
- *for no proper superset  $A'$  of  $A$ , the class  $\{C : B \subseteq C \subseteq A'\}$  is linearly ordered.*

**Proof** is similar to the proof of Proposition 2.

**Final Result.** Since we have subsethood, we also have union: the union of  $A_\alpha$  is the  $\subseteq$ -smallest set that contains all  $A - \alpha$ . We can thus define type-1 fuzzy sets as unions of 1-element type-1 fuzzy sets. Once we can detect type-1 fuzzy sets, we can use techniques from the previous section to detect crisp sets. Thus, *we can indeed detect type-1 fuzzy sets and crisp sets based only on subsethood relation between interval-valued fuzzy sets.*

## References

1. Klir, G., Yuan, B.: Fuzzy Sets and Fuzzy Logic. Prentice Hall, Upper Saddle River (1995)
2. Nguyen, H.T., Walker, E.A.: A First Course in Fuzzy Logic. Chapman and Hall/CRC, Boca Raton (2006)
3. Zadeh, L.A.: Fuzzy sets. Inf. Control **8**, 338–353 (1965)