**Notes on Intuitionistic Fuzzy Sets**
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# Intuitionistic fuzzy logic is not always equivalent to interval-valued one

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**Abstract:** It has been shown that from the purely mathematical viewpoint, the (traditional) intuitionistic fuzzy logic is equivalent to interval-valued fuzzy logic. In this paper, we show that if we go beyond the traditional "and"- and "or"-operations, then intuitionistic fuzzy logic becomes more general than the interval-valued one.

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## 1 Intuitionistic fuzzy logic, interval-valued fuzzy logic, and their equivalence: a brief reminder

**What we plan to do.** In this paper, we analyze the relation between intuitionoistic and interval-valued fuzzy logics. In this section, we remind the readers what are these two logics, and in what sense they are equivalent. After that, in Section 2, we explain that if we go beyond the traditional "and"- and "or"-operations, then these two logics are not always equivalent to each other: actually, the intuitionistic fuzzy logic becomes more general.

**Intuitionistic fuzzy logic: a brief reminder.** In the traditional fuzzy logic (see, e.g., [4, 10, 12]), for each statement $S$, we store the degree $d(S) \in [0, 1]$ to which we believe this statement to be true. In particular, for statements like "$x$ is small" corresponding to different values $x$, the corresponding degrees of belief $\mu(x)$ form a function known as the *membership function*.

In the traditional fuzzy logic, the degree of confidence in the negation $\neg S$ is estimated as $d(\neg S) = 1 - d(S)$.

This approach has a limitation: it does not allow us to distinguish between the situation when we know practically nothing about a statement $S$ and the situation when we have several arguments for $S$ and a similar number of arguments against $S$. In both cases, we do not have any reason to prefer $S$ or $\neg S$. So, in the traditional fuzzy logic, in both cases, we select $d(S)$ for which $d(S) = d(\neg S)$, i.e., for which $d(S) = 0.5$.

To take this difference into account, K. Atanassov proposed to separately store two degrees for each statement $S$: the degree $d^+(S) = d(S)$ to which the statement $S$ is confirmed and the degree $d^-(S) \stackrel{\text{def}}{=} d(\neg S)$ to which the negation of $S$ is confirmed; see, e.g., [1, 2, 3]. In these terms, the two above situations are different:

- the situation in which we know nothing about $S$ or $\neg S$ can be described as setting $d(S) = d(\neg S) = 0$; while
- the situation in which we have many arguments in favor of $S$ and many arguments in favor of $\neg S$ can be described by setting $d(S) = d(\neg S) > 0$ – e.g., by setting $d(S) = d(\neg S) = 0.5$.

The only requirement on these two numbers is that overall, they should not exceed 1: $d^+(S) + d^-(S) \leq 1$. Intuitively, this means that we can have arguments for $S$ and arguments against $S$, but these arguments should not be in contradiction to each other, they should be consistent with each other.

For statements like "$x$ is small", we thus get, for each value $x$, two degrees $\mu^+(x)$ and $\mu^-(x)$ for which $\mu^+(x) + \mu^-(x) \leq 1$. Due to the analogy with intuitionistic logic, in which the negation of a statement does not uniquely determine this statement, Atanassov called this idea *intuitionistic fuzzy logic*.

**Standard operations in intuitionistic fuzzy logic.** In the traditional $[0, 1]$-based fuzzy logic, the most widely used "and" and "or"-operations are $\min(a, b)$ and $\max(a, b)$. Because of this, Atanassov defined the standard "and"- and "or"-operations in intuitionistic fuzzy logic as follows:

- a belief in $S \,\&\, S'$ means that we believe in $S$ *and* we believe in $S'$; thus, if our degree of belief in $S$ is equal to $d^+(S)$ and our degree of belief in $S'$ is equal to $d^+(S')$, then it is reasonable to gauge our degree of belief in $S \,\&\, S'$ as $d^+(S \,\&\, S') = \min(d^+(S), d^+(S'))$;
- on the other hand, a belief in $\neg(S \,\&\, S')$, i.e., in $\neg S \vee \neg S'$ means that we either believe in $\neg S$ *or* we believe in $\neg S'$; thus, if our degree of belief in $\neg S$ is equal to $d^-(S)$ and our degree of belief in $\neg S'$ is equal to $d^-(S')$, then it is reasonable to gauge our degree of belief in $\neg(S \,\&\, S')$ as $d^-(S \,\&\, S') = \max(d^-(S), d^-(S'))$.

So, if our belief in $S$ is characterized by a pair $(d^+(S), d^-(S))$ and our belief in $S'$ is characterized by a pair $(d^+(S'), d^-(S'))$, then for $S \,\&\, S'$, we get

$$(d^+(S \,\&\, S'), d^-(S \,\&\, S')) = (\min(d^+(S), d^+(S')), \max(d^-(S), d^-(S'))).$$

Similarly, for $S \vee S'$, we get

$$(d^+(S \vee S'), d^-(S \vee S')) = (\max(d^+(S), d^+(S')), \min(d^-(S), d^-(S'))).$$

For negation, of course, the situation is even simpler: our degree of belief in $\neg S$ is simply equal to $d^-(S)$, and our degree of belief in the negation of $\neg S$ (i.e., in $S$ itself) is eqqual to $d^+(S)$:

$$(d^+(\neg S), d^-(\neg S)) = (d^-(S), d^+(S)).$$

**Interval-valued fuzzy logic: a reminder.** Interval-valued fuzzy logic is another way to make the distinction between the two above cases. In the interval-values fuzzy logic, instead of always assigning a single degree $d(S)$ to each statement, we allow, in situations when an expert is not sure about his or her degree, to assign an *interval* $[\underline{d}(S), \overline{d}(S)]$ of possible values of this degree; see, e.g., [6, 7, 9].

This enables us to take into account the above difference:

- in situations when we know nothing about a statement $S$, instead of selecting one value from the interval $[0, 1]$, we honestly say that we do not know what degree to assign, i.e., we assign the whole interval $[\underline{d}(S), \overline{d}(S)] = [0, 1]$ of possible values;
- on the other hand, in a situation in which we know a lot of arguments for $S$ and a lot of arguments against $S$, we can assign a single number $d(S) = 0.5$, i.e., a "degenerate" interval $[\underline{d}(S), \overline{d}(S)] = [0.5, 0.5]$ that consists of a single point 0.5.

Traditional "and"- and "or"-operations (and negation) can be naturally extended to the interval-valued case. Indeed, if we have intervals $[\underline{d}(S), \overline{d}(S)]$ and $[\underline{d}(S'), \overline{d}(S')]$ describing our belief in $S$ and $S'$, this means that our actual degrees of belief could be any numbers $d(S)$ and $d'(S)$ from these intervals. For each two such values, the degree of belief in $S \,\&\, S'$ can be computed as $\min(d(S), d(S'))$. Thus, to describe our confidence in $S \,\&\, S'$, it is reasonable to take the set of all such values $\min(d(S), d(S'))$ corresponding to all possible combinations of $d(S)$ and $d(S')$:

$$\{\min(d(S), d(S')) : d(S) \in [\underline{d}(S), \overline{d}(S)] \text{ and } d(S') \in [\underline{d}(S'), \overline{d}(S')]\}.$$

Since the minimum is a monotonic function of both its variables:

- its minimum is attained when both $d(S)$ and $d(S')$ attains their smallest possible values $\underline{d}(S)$ and $\underline{d}(S')$, and
- its maximum is attained when both $d(S)$ and $d(S')$ attains their largest possible values $\underline{d}(S)$ and $\underline{d}(S')$.

Thus, the above set of possible values is the interval

$$[\min(\underline{d}(S),\underline{d}(S')),\min(\overline{d}(S),\overline{d}(S'))].$$

This interval can be viewed as the result of applying the appropriate "and"-operation to the intervals $[\underline{d}(S),\overline{d}(S)]$ and $[\underline{d}(S'),\overline{d}(S')]$ describing the expert's degree of belief in $S$ and $S'$.

Similarly, for the "or"-operation, as the degree of belief in the disjunction $S \vee S'$, we take the set

$$\{\max(d(S),d(S')) : d(S) \in [\underline{d}(S),\overline{d}(S)] \text{ and } d(S') \in [\underline{d}(S'),\overline{d}(S')]\},$$

which, due to the monotonicity of $\max(a,b)$, takes the form of the interval

$$[\max(\underline{d}(S),\underline{d}(S')),\max(\overline{d}(S),\overline{d}(S'))].$$

For negation, we similarly have

$$\{1-d(S) : d(S) \in [\underline{d}(S),\overline{d}(S)]\} = [1-\overline{d}(S),1-\underline{d}(S)].$$

**If we use $\min$ and $\max$, then intuitionistic and interval-valued fuzzy logics are mathematically equivalent.** In intuitionistic fuzzy logic, we have some arguments $d^+(S)$ in favor of $S$, some arguments $d^-S(S)$ in favor of $\neg S$, and we also have uncertainty – which can be described by the remaining degree $1-d^+(S)-d^-(S)$. As we learn more about the statement $S$, this uncertainty may be replaced with arguments in favor of $S$ and/or with arguments in favor of $\neg S$.

The worst-case scenario, when we end up with the smallest possible confidence in $S$, is when all the uncertainty is replaced with arguments in favor of the negation $\neg S$. In this case, the resulting degree of belief in $S$ is equal to the original value $d^+(S)$.

The best-case scenario, when we end up with the largest possible confidence in $S$, is when all the uncertainty is replaced with arguments in favor of $S$. In this case, the resulting degree of belief in $S$ is equal to $d^+(S)=(1-d^+(S)-d^-(S))=1-d^-(S)$.

Thus, in the intuitionistic fuzzy case, the eventual degree of certainty can take any value from the interval $[d^+(S),1-d^-(S)]$. It is therefore reasonable to associate each intuitionistic fuzzy degree $(d^+(S),d^-(S))$ with the interval $[\underline{d}(S),\overline{d}(S)]=[d^+(S),1-d^-(S)]$.

Vice versa, if we know the values $\underline{d}(S)$ and $\overline{d}(S)$, we can easily find the values $d^+(S)$ and $d^-(S)$ for which $\underline{d}(S)=d^+(S)$ and $\overline{d}(S)=1-d^-(S)$: namely, we can take $d^+(S)=\underline{d}(S)$ and $d^-(S)=1-\overline{d}(S)$.

It turns out that this transformation preserves the above "and"-, "or"-, and "not"-operations. For example, if start with the degrees $(d^+(S),d^-(S)$ and $(d^+(S'),d^-(S'))$, then we can can compute our degree of belief in $S \,\&\, S'$ in two different ways.

First, we can directly apply the rules of intuitionistic fuzzy logic and get the pair

$$(d_1(S \,\&\, S'),d_1^-(S \,\&\, S')=(\min(d^+(S),d^+(S')),\min(d^-(S),d^-(S'))).$$

Alternatively, we can:

- transform the two degrees into intervals $[\underline{d}(S),\overline{d}(S)]=[d^+(S),1-d^-(S)]$ and $[\underline{d}(S'),\overline{d}(S')]$ $=[d^+(S'),1-d^-(S')]$;

- apply the "and"-operation of interval-valued fuzzy logic to get

$$[\underline{d}(S\,\&\,S'), \overline{d}(S\,\&\,S')] = [\min(\underline{d}(S), \underline{d}(S')), \min(\overline{d}(S), \overline{d}(S'))] =$$

$$[\min(d^+(S), d^+(S')), \min(1-d^-(S), 1-d^-(S'))];$$

- and then transform this interval back into the intuitionistic fizzy value

$$(d_2^+(S\,\&\,S'), d_2^-(S\,\&\,S')) =$$

$$(\min(d^+(S), d^+(S')), 1-\min(1-d^-(S), 1-d^-(S'))).$$

One can show that these two approaches lead to the same result. Indeed, we can immediately see that:

$$d_1(S\,\&\,S') = d_2(S\,\&\,S') = \min(d^+(S), d^+(S')).$$

For the negative degrees, we have $d_1^-(S\,\&\,S') = \min(d^-(S), d^-(S'))$ and $d_2^-(S\,\&\,S')) = 1 - \min(1-d^-(S), 1-d^-(S'))$. Since the function $1-x$ is decreasing, the smallest value of $1-x$ is attained when $x$ is the largest, i.e., in this case:

$$\min(1-d^-(S), 1-d^-(S')) = 1-\max(d^-(S), d^-(S')).$$

Thus,

$$d_2^-(S\,\&\,S') = 1-\min(1-d^-(S), 1-d^-(S')) =$$

$$1-(1-\max(d^-(S), d^-(S'))) = \max(d^-(S), d^-(S')).$$

So, we indeed have $d_1^-(S\,\&\,S') = d_2^-(S\,\&\,S')$.

Similar results hold for "or"-operations and for negation: we get the exact same results whether we use operations from the intuitionostic fuzzy logic or from interval-valued fuzzy logic. In this sense, these two logics are mathematically equivalent – although their semantics is different.

**What we plan to show.** The above equivalence results is based on using $\min$ and $\max$. $\min$ and $\max$ are the simplest possible "and"- and "or"-operations. In many applications, it is beneficial to consider different "and"- and "or"-operations. In this paper, we show that if we use different "and'– and "or"-operations, then, in general, the intuitionistic and interval-valued fuzzy logics are no longer mathematically equivalent.

# 2 In the general case, intuitionistic fuzzy logic and interval-valued fuzzy logic are not equivalent

**Let us consider the general case.** In the general case, we can use an arbitrary pair of an "and"-operation (t-norm) $f_\&(a,b)$ and an "or"-operation (t-conorm) $f_\vee(a,b)$.

In some cases, we use *dual* operations, for which $f_\vee(a,b) = 1-f_\&(1-a,1-b)$ and therefore $f_\&(a,b) = 1-f_\vee(1-a,1-b)$, but this is not always the best choice. For example, if we are looking for the smoothest possible control, then it makes sense to use a non-dual pair $f_\&(a,b) = a \cdot b$ and $f_\vee(a,b) = \max(a,b)$. Similarly, if we are looking for the most stable control, then the optimal selection is another non-dual pair $f_\&(a,b) = \min(a,b)$ and $f_\vee(a,b) = \min(a+b,1)$; see, e.g., [5, 8, 11].

Let us analyze what happens if we apply such a pair in both intuitionistic and interval-valued fuzzy cases.

**Applying a general pair or "and"- and "or"-operations to the intuitionostic fuzzy case.** Let us start this analysis with the case of "and". As we have mentioned, a belief in $S \,\&\, S'$ means that we believe in $S$ *and* that we believe in $S'$.

Thus, if our degree of belief in $S$ is equal to $d^+(S)$ and our degree of belief in $S'$ is equal to $d^+(S')$, then it is reasonable to gauge our degree of belief in $S \,\&\, S'$ as $d^+(S \,\&\, S') = f_\&(d^+(S), d^+(S'))$.

On the other hand, a belief in $\neg(S \,\&\, S')$, i.e., in $\neg S \vee \neg S'$ means that we either believe in $\neg S$ *or* we believe in $\neg S'$. Thus, if our degree of belief in $\neg S$ is equal to $d^-(S)$ and our degree of belief in $\neg S'$ is equal to $d^-(S')$, then it is reasonable to gauge our degree of belief in $\neg(S \,\&\, S')$ as $d^-(S \,\&\, S') = f_\vee(d^-(S), d^-(S'))$. So, we get

$$(d^+(S \,\&\, S'), d^-(S \,\&\, S')) = (f_\&(d^+(S), d^+(S')), f_\vee(d^-(S), d^-(S'))).$$

Similar, for "or", we get

$$(d^+(S \vee S'), d^-(S \vee S')) = (f_\vee(d^+(S), d^+(S')), f_\&(d^-(S), d^-(S'))).$$

**Applying a general pair or "and"- and "or"-operations to the interval-valued fuzzy case.** If we have intervals $[\underline{d}(S), \overline{d}(S)]$ and $[\underline{d}(S'), \overline{d}(S')]$ describing our belief in $S$ and $S'$, this means that our actual degrees of belief could be any numbers $d(S)$ and $d'(S)$ from these intervals. For each two such values, the degree of belief in $S \,\&\, S'$ can be computed as $f_\&(d(S), d(S'))$. Thus, to describe our confidence in $S \,\&\, S'$, it is reasonable to take the set of all such values $f_\&(d(S), d(S'))$ corresponding to all possible combinations of $d(S)$ and $d(S')$:

$$\{f_\&(d(S), d(S')) : d(S) \in [\underline{d}(S), \overline{d}(S)] \text{ and } d(S') \in [\underline{d}(S'), \overline{d}(S')]\}.$$

Since $f_\&(a,b)$ is a monotonic function of both its variables:

- its minimum is attained when both $d(S)$ and $d(S')$ attains their smallest possible values $\underline{d}(S)$ and $\underline{d}(S')$, and
- its maximum is attained when both $d(S)$ and $d(S')$ attains their largest possible values $\underline{d}(S)$ and $\underline{d}(S')$.

Thus, the above set of possible values is the interval

$$[f_\&(\underline{d}(S), \underline{d}(S')), f_\&(\overline{d}(S), \overline{d}(S'))].$$

Similarly, for the "or"-operation, as the degree of belief in the disjunction $S \vee S'$, we take the set

$$\{f_\vee(d(S), d(S')) : d(S) \in [\underline{d}(S), \overline{d}(S)] \text{ and } d(S') \in [\underline{d}(S'), \overline{d}(S')]\},$$

which, due to the monotonicity of an "or"-operation $f_\vee(a,b)$, takes the form of the interval

$$[f_\vee(\underline{d}(S),\underline{d}(S')),f_\vee(\overline{d}(S),\overline{d}(S'))].$$

**Comparing the results.** Once we know the intuionistic fuzzy degrees $(d^+(S),d^-(S))$ and $(d^+(S'),d^-(S'))$ for two statements $S$ and $S'$, then one way to get the degree for $S\,\&\,S'$ is to apply the formulas described for the intuitionistic fuzzy case:

$$(d_1^+(S\,\&\,S'),d_1^-(S\,\&\,S'))=(f_\&(d^+(S),d^+(S')),f_\vee(d^-(S),d^-(S'))).$$

Alternatively, we can:

- transform the two degrees into intervals $[\underline{d}(S),\overline{d}(S)]=[d^+(S),1-d^-(S)]$ and $[\underline{d}(S'),\overline{d}(S')]$ $=[d^+(S'),1-d^-(S')]$;
- apply the "and"-operation of interval-valued fuzzy logic to get

$$[\underline{d}(S\,\&\,S'),\overline{d}(S\,\&\,S')]=[f_\&(\underline{d}(S),\underline{d}(S')),f_\&(\overline{d}(S),\overline{d}(S'))]=$$

$$[f_\&(d^+(S),d^+(S')),f_\&(1-d^-(S),1-d^-(S'))];$$

- and then transform this interval back into the intuitionistic fizzy value

$$(d_2^+(S\,\&\,S'),d_2^-(S\,\&\,S'))=$$

$$(f_\&(d^+(S),d^+(S')),1-f_\&(1-d^-(S),1-d^-(S'))).$$

By comparing the degrees $d_1(S,S')=(d_1^+(S\,\&\,S'),d_1^-(S\,\&\,S'))$ and $d_2(S,S')=(d_2^+(S\,\&\,S'),$ $d_2^-(S\,\&\,S'))$, we can see that the positive degrees are still the same $d_1^+(S\,\&\,S')=d_2^+(S\,\&\,S')$, while the negative degrees are equal only when

$$d_1^-(S\,\&\,S')=f_\vee(d^-(S),d^-(S'))=$$

$$1-f_\&(1-d^-(S),1-d^-(S'))=d_2^-(S\,\&\,S').$$

This condition is always satisfied only if

$$f_\vee(a,b)=1-f_\&(1-a,1-b),$$

i.e., only if the "and"- and "or"-operations are dual. In all other cases, the the "and"-operations of the intuitionsitic fuzzy logic and of the interval-valued fuzzy logic are *not* equivalent.

Similarly, for the "or"-operations, if we directly apply the formulas of the intuitionistic fuzzy logic, we get

$$(d_1^+(S\vee S'),d_1^-(S\vee S'))=(f_\vee(d^+(S),d^+(S')),f_\&(d^-(S),d^-(S'))).$$

On the other hand, if we:

- transform the two degrees into intervals $[\underline{d}(S),\overline{d}(S)]=[d^+(S),1-d^-(S)]$ and $[\underline{d}(S'),\overline{d}(S')]$ $=[d^+(S'),1-d^-(S')]$;

- apply the “or”-operation of interval-valued fuzzy logic to get

$$[\underline{d}(S \vee S'), \overline{d}(S \vee S')] = [f_\vee(\underline{d}(S), \underline{d}(S')), f_\vee(\overline{d}(S), \overline{d}(S'))] =$$

$$[f_\vee(d^+(S), d^+(S')), f_\vee(1 - d^-(S), 1 - d^-(S'))];$$

- and then transform this interval back into the intuitionistic fizzy value, we get

$$(d_2^+(S \vee S'), d_2^-(S \vee S')) =$$

$$(f_\vee(d^+(S), d^+(S')), 1 - f_\vee(1 - d^-(S), 1 - d^-(S'))).$$

By comparing the degrees $d_1(S, S') = (d_1^+(S \vee S'), d_1^-(S \vee S'))$ and $d_2(S, S') = (d_2^+(S \vee S'), d_2^-(S \vee S'))$, we can see that the positive degrees are the same $d_1^+(S \vee S') = d_2^+(S \vee S')$, but the negative degrees are equal only when

$$d_1^-(S \vee S') = f_\&(d^-(S), d^-(S')) =$$

$$1 - f_\vee(1 - d^-(S), 1 - d^-(S')) = d_2^-(S \vee S').$$

This condition is always satisfied only if

$$f_\&(a, b) = 1 - f_\vee(1 - a, 1 - b),$$

i.e., only if the “and”- and “or”-operations are dual – the same condition as for the “and”-operation. In all other cases, the “or”-operations of the intuitionsitic fuzzy logic and of the interval-valued fuzzy logic are *not* equivalent.

# 3 Which pairs of “and”- and “or”-operations lead to intuitionsitic fuzzy sets?

**Formulation of the problem.** Now that we know that for different pairs of “and”- and “or”-operations we get results which are not equivalent to interval-valued fuzzy sets, a natural question is: which pairs of operations are allowed? When is the result of applying these operations always an intuitionstic fuzzy set?

**Re-wording the corresponding requirement in precise terms.** For two statements $S$ and $S'$, we get $(d^+(S \,\&\, S'), d^-(S \,\&\, S')) = (f_\&(d^+(S), d^+(S')), f_\vee(d^-(S), d^-(S')))$. For this pair to form an intuionistic fuzzy set, we need to make sure that

$$f_\&(d^+(S), d^+(S')) + f_\vee(d^-(S), d^-(S')) \le 1.$$

Let us denote $a \stackrel{\text{def}}{=} d^+(S)$ and $b \stackrel{\text{def}}{=} d^+(S')$. Once we know $d^+(S) = a$, the only restriction of $d^-(S)$ is that $d^+(S) + d^-(S) \le 1$, so $d^-(S)$ can take any value from 0 to $1 - a$. Similarly, once we know $d^+(S') = b$, then $d^-(S')$ can take any value from 0 to $1 - b$.

We want the above inequality to be satisfied for all possible values $d^-(S)$. This is equivalent to requiring that the largest possible value of the left-hand side of this inequality to be smaller than or equal to 1. An "or"-operation is increasing in both variables, so its largest value is attained when both its arguments are the largest, i.e., when $d^-(S) = 1 - a$ and $d^-(S') = 1 - b$. Thus, the requirement that for given $a = d^+(S)$ and $b = d^+(S')$, the above inequality holds for all possible values of $d^-(S)$ and $d^-(S')$ is equivalent to requiring that

$$f_\&(a, b) + f_\vee(1 - a, 1 - b) \le 1.$$

So, we need to make sure that this inequality holds for all $a$ and $b$.

Similarly, we have $(d^+(S \vee S'), d^-(S \vee S')) = (f_\vee(d^+(S), d^+(S')), f_\&(d^-(S), d^-(S')))$, so for "or"-operation, the requirement that we always have an intuitionistic fuzzy set is equivalent to

$$f_\vee(d^+(S), d^+(S')) + f_\&(d^-(S), d^-(S')) \le 1.$$

If we fix $a = d^-(S)$ and $b = d^-(S')$, then possible values of $d^+(S)$ and $d^+(S')$ can take any values from 0 to, correspondingly, $1-a$ and $1-b$. Thus, the requirement that the above inequality always holds is equivalent to $f_\&(a, b) + f_\vee(1 - a, 1 - b) \le 1$.

Interestingly, this is exactly the same requirement as for the "and"-operation, so we conclude that the requirement that the "and"-operation always returns an intuitionsitic fuzzy set is equivalent to the requirement that the "or"-operation always returns the intuitionstic fuzzy set. In both cases, we need the same inequality to hold for all $a$ and $b$:

$$f_\&(a, b) + f_\vee(1 - a, 1 - b) \le 1.$$

**Analysis of this requirement.** The above requirement can be equivalently reformulated as

$$f_\&(a, b) \le 1 - f_\vee(1 - a, 1 - b).$$

We have already mentioned that for every "or"-operation $f_\vee(a, b)$, the expression $f^*_\&(a, b) \stackrel{\text{def}}{=} 1 - f_\vee(1 - a, 1 - b)$ is an "and"-operation; this "and"-operation is called *dual*. Thus, the above requirement is equivalent to stating that that for all $a$ and $b$, we have

$$f_\&(a, b) \le f^*_\&(a, b),$$

where $f_\&(a, b)$ is the original "and"-operation and $f^*_\&(a, b)$ is an "and"-operation which is dual to the original "or"-operation $f_\vee(a, b)$.

Such pair of "and"-operations are common: e.g., $a \cdot b \le \min(a, b)$ for all $a$ and $b$.

**Case of Archimedean "and"-operations.** Each Archimedean "and"-operation has the form $f_\&(a, b) = f^{-1}(f(a) + f(b))$ for some increasing function $f(x)$; see, e.g., [4, 10]. For example, $a \cdot b = \exp(\ln(a) + \ln(b))$.

If the dual operation is also Archimedean, then $f^*_\&(a, b) = g^{-1}(g(a) + g(b))$ for some increasing function $g(x)$. In these terms, the above requirement takes the form

$$f^{-1}(f(a) + f(b)) \le g^{-1}(g(a) + g(b)).$$

The left-hand side $c$ of this inequality comes from the requirement that $f(a) + f(b) = f(c)$. For this value, we have $g(a) + g(b) = g(z)$, where by $z$, we denoted the right-hand side of the desired inequality. Since the function $g(x)$ is monotonic, the condition that $c \le z$ is equivalent to $g(c) \le g(z)$. So, we require that if $f(a) + f(b) = f(c)$, then $g(c) \le g(a) + g(b)$.

Let us denote $A \stackrel{\text{def}}{=} f(a)$, $B \stackrel{\text{def}}{=} f(b)$, and $C \stackrel{\text{def}}{=} f(c) = A + B$, and let us denote $h(x) \stackrel{\text{def}}{=} g(f^{-1}(x))$. Then, $h(A) = g(f^{-1}(A)) = g(a)$; similarly $h(B) = g(b)$, and $h(C) = h(A + B) = g(c)$.

In these terns, the above inequality means that for all possible values of $A$ and $B$, we have

$$h(A + B) \le h(A) + h(B).$$

This property is well-known in mathematics: functions $h(x)$ satisfying this property are known as *sub-additive*.

So, to satisfy the above requirement for Archmedean operations, it is necessary and sufficient to make sure that the auxiliary function $h(x) = g(f^{-1}(x))$ connecting the generating functions $f(x)$ and $g(x)$ of the two "and"-operations $f_\&(a, b)$ and $f^*_\&(a, b)$ is sub-additive.

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